

LEVI-FLAT FILLING OF REAL TWO-SPHERES IN SYMPLECTIC MANIFOLDS (II)

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Abstract. We consider a compact almost complex manifold (M, J, ω) with smooth Levi convex boundary ∂M and a tame symplectic form ω . Suppose that S^2 is a real two-sphere, containing complex elliptic and hyperbolic points and generically embedded into ∂M . We prove a result on filling S^2 by holomorphic discs.

Résumé. On considère une variété presque complexe (M, J, ω) avec la frontière Levi convexe ∂M et une tame forme sympléctique ω . Soit S^2 une 2-sphere réelle avec des points elliptiques et hyperboliques, plongée génériquement dans ∂M . On démontre un résultat sur le remplissage de S^2 par des disques holomorphes.

1. INTRODUCTION

This expository paper is the second part of [10]. We keep the same notations and terminology. Our main result is the following.

Theorem 1.1. *Let (M, J, ω) be a compact almost complex manifold of complex dimension 2 with a tame symplectic form ω and smooth boundary ∂M . Let also S^2 be a real 2-sphere embedded into ∂M . Assume that the following assumptions hold:*

- (i) *M contains no non-constant J -holomorphic spheres.*
- (ii) *the boundary ∂M of M is a smooth Levi convex hypersurface containing no non-constant J -holomorphic discs.*
- (iii) *S^2 has only elliptic and good hyperbolic complex points. Furthermore, ∂M is strictly Levi convex in a neighborhood of every hyperbolic point.*

Then after an arbitrarily small C^2 -perturbation near hyperbolic points there exists a smooth Levi-flat hypersurface $\Gamma \subset M$ with boundary S^2 . This hypersurface is foliated by J -holomorphic discs with boundaries attached to S^2 .

In [10] we studied the filling of a two-sphere containing only isolated elliptic complex points. The present work is devoted to the more general case where hyperbolic points occur. We impose on M and ∂M precisely the same assumptions as in [10].

Conditions (i)-(ii) are essential to extend a local filling of S^2 by boundaries of pseudoholomorphic discs, starting at an elliptic point, up to hyperbolic points without appearance of

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sphere or disc bubbles. Assumption (i) may be weakened (see the discussion in [10]), however (ii) is in general necessary (see the last section). Condition (iii) is a technical assumption but it is sufficient for consistent applications. A precise definition of a good hyperbolic point is given in the next section. We mention that by definition the almost complex structure J is supposed to be integrable near a good hyperbolic point and each hyperbolic point can be made a good one by an arbitrary small C^2 deformation of the sphere S^2 near this point. In the last section we include a detailed discussion and comparison of this result with known results obtained by several authors.

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2. GOOD HYPERBOLIC POINTS: BEDFORD-KLINGENBERG'S ANALYSIS

Denote by S_*^2 the set of totally real points in S^2 . Let f be a Bishop disc for S^2 . We call f *hyperbolic* if there exists a finite set of points $\zeta_j \in \partial\mathbb{D}$, $j = 1, \dots, k$, such that every point $f(\zeta_j)$ is a hyperbolic point of S^2 and $f(\zeta) \in S_*^2$ for every $\zeta \in \partial\mathbb{D} \setminus \{\zeta_1, \dots, \zeta_k\}$. We recall some properties of hyperbolic discs in a neighborhood of a hyperbolic point. These results were obtained by E. Bedford-W. Klingenberg [2] in the case where (M, J) coincides with a strictly pseudoconvex domain in (\mathbb{C}^2, J_{st}) . *Everywhere in this section we suppose that the almost complex structure J is integrable near every hyperbolic point of S^2 .* This assumption will be crucially used.

2.1. Good hyperbolic points. Let S^2 be a two-sphere generically embedded into an almost complex manifold (M, J) of complex dimension 2. In what follows it is convenient to view M as a smoothly bounded subdomain in an almost complex manifold \tilde{M} .

Suppose that p is a hyperbolic point of S^2 . Since the almost complex structure J is integrable in a neighborhood of p , there exist complex coordinates $(z, w) \in \mathbb{C}^2$ defined in a neighborhood of p such that in these coordinates the origin corresponds to p , $J = J_{st}$ in a neighborhood of the origin and S^2 is locally defined by the expression :

$$(2.1) \quad w = z\bar{z} + \gamma Re z^2 + o(|z|^2)$$

where $\gamma > 1$. Then we can represent the boundary ∂M near the origin in the form $\partial M = \{\rho = 0\}$ with

$$\rho(z, w) = -Re w + \alpha_1 Im w + |z|^2 + \alpha_2 |w|^2 + \gamma Re z^2 + Re[(\alpha_3 + i\alpha_4)z\bar{w}] + o(|z|^2);$$

here α_j are real constants. We will call such a ρ a *local defining function* of ∂M near a good hyperbolic point.

After an arbitrarily small C^2 -deformation near the origin S^2 can be transformed to the model quadric :

$$(2.2) \quad w = \psi(z) = z\bar{z} + \gamma Re z^2$$

which coincides with the initial sphere outside a neighborhood of p . This follows by multiplying the $o(|z|^2)$ -term in the right-hand side by a suitable cut-off function vanishing near the origin, see [2]. More precisely, denote the $o(|z|^2)$ -term in (2.1) by $\phi(z)$. Then we replace

$\phi(z)$ by $\chi(|z|/\varepsilon)\phi(z)$, where χ is a smooth function with $\chi(t) = 0$ for $t < 1$ and $\chi(t) = 1$ for $t > 2$. It is easy to see that the quantity $(1 - \chi(|z|/\varepsilon))\phi(z)$ tends to 0 in the C^2 -norm as $\varepsilon \rightarrow 0$.

The boundary ∂M also must be slightly deformed near the origin in order to contain the perturbed S^2 given by (2.2). Since the hypersurface ∂M is strictly Levi convex, it remains strictly Levi convex after a small C^2 -deformation. Thus, in the present paper we deal with hyperbolic points which can be written in the form (2.2). Throughout the rest of the paper we keep the notation ρ and ψ for the local defining functions of ∂M and S^2 respectively, near a good hyperbolic point, introduced above.

Now we describe an additional restriction on the values of the parameter γ imposed through the present paper; it considerably simplifies the study of hyperbolic discs. Consider the proper holomorphic map $H : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by

$$H(z, w) = (z, zw + \frac{\gamma}{2}(z^2 + w^2)).$$

Then H determines a two-fold branched covering of \mathbb{C}^2 . The pull-back $H^{-1}(S^2)$ in a neighborhood of the origin consists of two totally real subspaces $E_1 = \{w = \bar{z}\}$ and $E_2 = \{w = -\bar{z} - (2/\gamma)z\}$. Denote by τ_j the antiholomorphic involution with respect to E_j . It is easy to see that

$$\tau_1 = \tau \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\tau_2 = \tau \circ \begin{pmatrix} -2/\gamma & -1 \\ (2/\gamma)^2 - 1 & 2/\gamma \end{pmatrix}$$

where τ denotes the usual conjugation in \mathbb{C}^2 . Since the matrices in the expressions of τ_1 and τ_2 commute with τ , we may with some abuse of terminology view τ_j as elements of the group $GL(2, \mathbb{R})$.

Lemma 2.1. *There exists a dense subset Λ in $]0, +\infty[$ such that for every $\gamma \in \Lambda$, the involutions τ_1 and τ_2 generate a finite group isomorphic to the dihedral group D_{2n} for some integer n (in general, depending on γ).*

Proof. Every matrix

$$C = \begin{pmatrix} c_1 & c_2 \\ c_2 & c_1 \end{pmatrix}$$

commutes with τ_1 . Setting $a = 2/\gamma$, we have

$$\begin{aligned} \tilde{\tau}_2 &= C^{-1}\tau_2C \\ &= \frac{1}{c_1^2 - c_2^2} \begin{pmatrix} -ac_1^2 - a^2c_1c_2 - ac_2^2 & -c_1^2 - 2ac_1c_2 - (a^2 - 1)c_2^2 \\ (a^2 - 1)c_1^2 + 2c_1c_2a + c_2^2 & ac_1^2 + a^2c_1c_2 + ac_2^2 \end{pmatrix} \end{aligned}$$

Since the trace of $\tilde{\tau}_2$ is zero, it suffices to make it symmetric in order to assure that this matrix is orthogonal (perhaps, after a multiplication by a suitable diagonal matrix). The condition for $\tilde{\tau}_2$ to be symmetric is

$$ax^2 + 4x + a = 0$$

with $x = c_1/c_2$. This equation admits a real solution since $0 < a < 2$. The vector $\begin{pmatrix} \nu \\ 1 \end{pmatrix}$ generating the axis of reflection for $\tilde{\tau}_2$ satisfies

$$\tilde{\tau}_2 \begin{pmatrix} \nu \\ 1 \end{pmatrix} = \begin{pmatrix} \nu \\ 1 \end{pmatrix}$$

From this we deduce

$$\nu = \frac{-c_1^2 - 2ac_1 + 1 - a^2}{(1+a)c_1^2 + a^2c_1 + a - 1}.$$

This implies that ν is a non-constant algebraic function of the variable $\gamma = 2/a$.

The group generated by τ_1 and $\tilde{\tau}_2$ is generated by orthogonal reflections about the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \nu \\ 1 \end{pmatrix}$. If the angle between these vectors is a rational multiple of π of the form $n\pi/m$ with relatively prime numbers n and m , then $\tilde{\tau}_2$ and τ_1 generate the dihedral group D_{2m} . Since this occurs for a dense set $1 < \gamma < \infty$, we obtain the conclusion. Q.E.D.

Definition 2.2. Suppose that S^2 is contained in a strictly Levi convex hypersurface near a hyperbolic point $p \in S^2$. This point is called *good* if

- (i) J is integrable in a neighborhood of p ,
- (ii) there exist local holomorphic coordinates near p such that S^2 has the form (2.2),
- (iii) γ satisfies the conclusion of Lemma 2.1 i.e. $\gamma \in \Lambda$.

2.2. Analytic extension past a good hyperbolic point. Now we describe the behaviour of hyperbolic discs near a good hyperbolic point.

Lemma 2.3. *Suppose that S^2 is given by (2.2) near the origin which is a good hyperbolic point. Let U be a neighborhood of the origin in \mathbb{C}^2 and Y be a closed complex purely 1-dimensional variety in $U \setminus S^2$ (i.e. every irreducible component of Y has complex dimension 1, see [4] for more details). Then (possibly after shrinking U) Y is contained in a closed complex purely 1-dimensional subset X in U .*

Thus, a complex 1-dimensional analytic set Y extends analytically past the origin. This key result was proved in [2] under an additional assumption that Y is a graph of a holomorphic function continuous up to the boundary on a domain in \mathbb{C} whose boundary contains the origin. However the argument still works without this assumption.

Proof. The pull-back $V_0 = H^{-1}(Y)$ is a closed complex 1-dimensional subset in $H^{-1}(U \setminus S^2) = H^{-1}(U) \setminus (E_1 \cup E_2)$. Of course, an analytic subset defined outside a single real analytic totally real manifold can be extended through this manifold by the reflection principle [4]. Our case is more subtle since the standard reflection principle cannot be applied at the origin where the totally real planes intersect. So we first use the involutions σ_j and the reflection principle at points of $E_1 \cup E_2$ off the origin in order to obtain an analytic subset in a punctured neighborhood of the origin.

By Lemma 2.1 the group G generated by the reflections about E_1 and E_2 is finite. Set $\tilde{U} = H^{-1}(U)$. Then \tilde{U} is a neighborhood of the origin and $V := \cup_{\sigma \in G} \sigma(V_0)$ is a complex 1-dimensional subset in $\tilde{U} \setminus (\cup_{\sigma \in G} \sigma(E_1 \cup E_2))$. Since every $\sigma \in G$ is an antiholomorphic reflection with respect to E_1 or E_2 , the reflection principle for complex analytic varieties (see for instance [4]) implies that the closure \overline{V} of V is an analytic subset in $\tilde{U} \setminus \{0\}$. Then by the Remmert-Stein removal singularities theorem (see for instance [4]) the closure \overline{V} is a complex

analytic variety in \tilde{U} . Since H is proper, the image $H(\overline{V})$ is a complex 1-dimensional variety in U containing Y . Q.E.D.

Let $f : \mathbb{D} \rightarrow M$ be a non-constant map. Then $f(\mathbb{D})$ is contained in M , i.e. for every point $\zeta \in \mathbb{D}$ its image $f(\zeta)$ is not on the boundary ∂M . This is a consequence of the J -convexity of ∂M and of the assumption that the boundary ∂M contains no J -holomorphic discs, see [10]. Let E be a non-empty subset of $\partial \mathbb{D}$. The cluster set $C(f, E)$ of f on E is defined as the set of all points p such that there exists a sequence (ζ_n) in \mathbb{D} converging to a point $\zeta \in E$ with $f(\zeta_n) \rightarrow p$. A map $f : \mathbb{D} \rightarrow M$ is proper if and only if the cluster set $C(f, \partial \mathbb{D})$ is contained in ∂M (one can say that in this sense the boundary values of f belong to ∂M). Because of the above remark for a proper J -holomorphic map $f : \mathbb{D} \rightarrow M$ one has $C(f, \partial \mathbb{D}) = \overline{f(\mathbb{D})} \setminus f(\mathbb{D})$. Recall the well-known fact that the cluster set $C(f, \partial \mathbb{D})$ is connected. Indeed, if not, there are two disjoint open sets U and V such that $C(f, \partial \mathbb{D}) \subset U \cup V$ and $C(f, \partial \mathbb{D})$ meets each of the sets U and V . Then for $r < 1$ close enough to 1 the connected set $f(\{r < |\zeta| < 1\})$ is contained in $U \cup V$ and intersects each set U and V which is impossible.

Let now $f : \mathbb{D} \rightarrow M$ be a proper J -holomorphic map such that the cluster set $C(f, \partial \mathbb{D})$ is contained in S^2 . We will see in Section 4 that this condition always hold for a disc which is the limit of a sequence of Bishop's discs attached to the totally real part S_*^2 of the sphere S^2 .

Let $p \in S^2$ be a point of S^2 and $p \in C(f, \partial \mathbb{D})$. Consider an open neighborhood U_p of p . Choosing U_p small enough, we can take it in the form $U_p = \{\phi < 0\}$ where ϕ is a strictly J -plurisubharmonic function on U_p . It suffices to fix a local coordinate system (z_1, z_2) centered at p such that in these coordinates $J(0) = J_{st}$ and set $\phi(z) = |z_1|^2 + |z_2|^2 - \varepsilon$ with $\varepsilon > 0$ small enough. Denote by $B(p, \varepsilon) = \{\phi < 0\}$ the ball of radius ε centered at p . This construction may be performed for all points q of S^2 . It provides a family of neighborhoods (U_q) , $q \in U_q$, and a family of strictly J -plurisubharmonic functions on U_q , (ϕ_q) , depending smoothly on q , such that $\phi_p = \phi$. We use the notation $B(q, \varepsilon) = \{\phi_q < \varepsilon\}$.

Lemma 2.4. *For $\varepsilon > 0$ small enough each connected component of the intersection $f(\mathbb{D}) \cap B(p, \varepsilon)$ is a disc (more precisely, the image of a J -holomorphic disc).*

Proof. Fix $\varepsilon > 0$ small enough such that every ball $B(q, \varepsilon)$ is compactly contained in the neighborhood U_q of q chosen above, for all $q \in S^2$. Let G be an open connected component of the pull-back $f^{-1}(B(p, \varepsilon))$. Suppose by contradiction that G is not simply connected. If the closure \overline{G} is simply connected, the subharmonic function $\rho \circ f$ achieves its maximum (equal to ε) at an interior point of \overline{G} which gives a contradiction. Consider the case where \overline{G} is not simply connected i.e. G has at least one hole H with non-empty interior. Then $C = f(H)$ is a compact J -holomorphic curve with boundary $\partial C \subset B(p, \varepsilon)$. Since $C(f, \partial \mathbb{D})$ is contained in S^2 , we can choose ε small enough such that the curve C is compactly contained in the neighborhood $U = \cup_{q \in S^2} U_q$ of S^2 . On the other hand, the curve C is not contained in $B(p, \varepsilon)$. Then, considering the family of balls $B(q, \tau)$ and their translations along the real normals to ∂M , we can find by continuity suitable q and $\tau > 0$ such that the J -holomorphic curve C touches the boundary of the ball $B(q, \tau)$ from inside at some point a . But the boundary $\partial B(q, \tau)$ is strictly J -convex and admits a strictly plurisubharmonic defining function near a as described above. Then the restriction of this function on C is a subharmonic function admitting a local maximum at an interior point. This contradicts the maximum principle.

Thus, every open component of $f^{-1}(B(p, \varepsilon))$ is simply connected. Reparametrizing it by \mathbb{D} via the Riemann mapping theorem, we conclude. Q.E.D.

We point out that the above lemma claims that every single connected component of the intersection $f(\mathbb{D}) \cap B(p, \varepsilon)$ is a disc. *A priori*, such an intersection can have several connected components.

Let $p \in S^2$ be a good hyperbolic point. Consider a proper J -holomorphic map $f : \mathbb{D} \rightarrow M$ such that the point p belongs to the cluster set $C(f, \partial\mathbb{D}) \subset S^2$. Let U be a coordinate neighborhood of p provided by Definition 2.2; in particular, we identify p with the origin. Then $Y = f(\mathbb{D}) \cap U$ is a closed complex 1-dimensional subset in $U \setminus S^2$. By Lemma 2.4 this set consists of a finite number of holomorphic discs in $M \cap U$; we choose one of them and again denote it by Y . By Lemma 2.3 the variety Y extends as a complex 1-dimensional set \tilde{Y} past S^2 . By the uniqueness theorem for complex analytic sets [4] there exists a unique irreducible component of \tilde{Y} containing Y ; we still denote this component by \tilde{Y} . In particular, Y and \tilde{Y} do not contain the w -axis (if not, \tilde{Y} would coincide with the w -axis by the uniqueness theorem).

Let now $\pi : (z, w) \mapsto z$ be the canonical projection. Since \tilde{Y} does not contain the w -axis, the restriction $\pi|_{\tilde{Y}}$ is proper when U is small enough. More precisely, the intersection $\pi^{-1}(0) \cap \tilde{Y}$ is discrete near the origin and taking a neighborhood $U = U' \times U''$ small enough, U' and U'' being neighborhoods of the origin in \mathbb{C} , we obtain that $\pi : \tilde{Y} \cap U \rightarrow U'$ is proper (see [4]).

Then Y is the graph $\{w = g(z)\}$ of a function g holomorphic in a domain D in \mathbb{C} , and the boundary of D contains the origin. Since the variety Y extends analytically past S^2 , the function g is continuous up to the boundary of D and extends past the origin as a multivalued complex analytic function. Furthermore, g has a Puiseux expansion at the origin. Since $g(z) = \psi(z)$ on ∂D , we have $g(z) = O(z^2)$ for $z \in \partial D$ and all terms of this expansion are $O(z^2)$. Then there exists a positive integer m such that in a neighborhood of the origin g is given in D by the Puiseux expansion

$$(2.3) \quad g(z) = \sum_{k \geq 2m} g_k z^{k/m}.$$

The representation (2.3) gives much useful information about the behaviour of a hyperbolic disc near a good hyperbolic point. For instance, we obtain that the boundary of $f(\mathbb{D})$ is a continuous curve (with a finite number of real analytic components intersecting at the origin) near the origin. Now under the assumption that the area of f is bounded (which always holds in our situation) one can easily show that the map f itself is necessarily continuous on the closed disc $\overline{\mathbb{D}}$. The proof is based on a classical argument from the one-variable theory of conformal maps. This justifies the terminology “a hyperbolic disc” since by definition such a disc is a Bishop disc and so is continuous up to the boundary. We postpone the proof of this fact to Section 4, but in the rest of this section one may assume that a hyperbolic disc is continuous on $\overline{\mathbb{D}}$.

2.3. Approaching a good hyperbolic point by a disc. Following [2] we study a local behaviour of hyperbolic discs near a good hyperbolic point. Since these results will be crucially used, for reader’s convenience we include the proofs.

We begin with a more precise information about the Puiseux expansion (2.3):

Lemma 2.5. *In the expansion (2.3) the sum is taken over the set of integers k satisfying $k > 2m$.*

Proof. Suppose by contradiction that g_{2m} in (2.3) does not vanish. Then

$$(2.4) \quad g(z) = az^2 + O(z^{2+1/m}),$$

with $a \neq 0$. Recall that g is holomorphic on a domain D in \mathbb{C} whose boundary contains the origin. Denote by \mathbb{D}^+ the upper half-disc $\mathbb{D}^+ = \{\zeta \in \mathbb{C} : |\zeta| < 1, \operatorname{Im} \zeta > 0\}$. It is convenient to assume that the hyperbolic disc f under consideration is defined and holomorphic on \mathbb{D}^+ , continuous on $\overline{\mathbb{D}^+}$ and $f([-1, 1]) \subset S^2$ i.e. $f(\mathbb{D}^+)$ is glued to the sphere S^2 along the real segment $[-1, 1] = \{\zeta : -1 \leq \operatorname{Re} \zeta \leq 1, \operatorname{Im} \zeta = 0\}$; furthermore, $f(0) = 0$. One can always achieve these conditions, reparametrizing f by a suitable conformal isomorphism. The image $f(\mathbb{D}^+)$ coincides with the graph of g over the domain D . As above, denote by $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$, $\pi(z, w) = z$, the canonical projection. If $\varepsilon > 0$ is small enough, the intersection $\partial D \cap \varepsilon \mathbb{D}$ consists of two real curves $\gamma_+ = (\pi \circ f)([-1, 0])$ and $\gamma_- = (\pi \circ f)([0, 1])$. Since S^2 is given by (2.2) near the origin, $\gamma_+ \cup \gamma_-$ is contained in $\{z : \operatorname{Im} g(z) = 0\}$. Hence γ_{\pm} are real analytic 1-dimensional sets. By (2.4) each γ_{\pm} is tangential at the origin to one of the real lines satisfying the equation $\{\operatorname{Im} az^2 = 0\}$. Therefore, the domain D at the origin is asymptotic to an angle of size κ (i.e. the curves γ_{\pm} are tangent at the origin to the rays bounding this angle) which is a non-zero integer multiple of $\pi/2$. Consider separately the possible cases.

Case 1: $\kappa \geq \pi$. Recall that in a neighborhood U of the origin the disc $f(\mathbb{D}^+)$ is contained in the domain $M \cap U = \{\rho < 0\}$ where a strictly plurisubharmonic function ρ is a local defining function for M introduced in Subsection 2.1. The composition $\phi(z) = \rho(z, g(z))$ is a negative subharmonic function on D and its gradient vanishes at the origin. By assumption on κ , one can find an open disc G contained in D such that $\partial G \cap \partial D = \{0\}$. Applying to ϕ the Hopf lemma on G , we obtain that $\partial\phi(0)/\partial z \neq 0$ which is a contradiction.

Case 2: $\kappa = \pi/2$. Since $\operatorname{Im} g(z) = 0$ for $z \in \gamma_+ \cup \gamma_-$, we conclude that

$$g(z) = \pm |az^2| + O(|z|^{2+1/m}), z \in \gamma_+ \cup \gamma_-$$

and furthermore, g has opposite signs on γ_+ and γ_- . Every curve γ_{\pm} is tangent at the origin to a line forming an angle μ_{\pm} with the axis x , so that $\mu_- = \mu_+ + \pi/2$. The function ψ (defined in (2.2) in the polar coordinates $z = re^{i\theta}$) has the form $\psi(z) = r^2 \chi(\theta)$ with $\chi(\theta) = 1 + \gamma \cos(2\theta)$. Consider a sequence $z_k = r_k e^{i\theta_k}$ in γ_+ converging to 0. Then for every k one has $\psi(e^{i\theta_k}) = r_k^{-2} g(r_k e^{i\theta_k})$. Passing to the limit as $k \rightarrow \infty$, we obtain that $\psi(e^{i\mu_+})$ is equal to α or $-\alpha$. Repeating the same argument for γ_- , we obtain $\psi(e^{i\mu_+}) = -\psi(e^{i\mu_+ + \pi/2})$. But this means that μ_+ satisfies the equation

$$1 + \gamma \cos(2\mu_+) = -1 - \gamma \cos(2(\mu_+ + \pi/2)).$$

But this equation does not admit any solution. Thus, none of the above cases can occur. This implies $a = 0$. Q.E.D.

Let D^+ and D^- denote the connected components of $\{z \in \mathbb{C} : z\bar{z} + \gamma \operatorname{Re} z^2 < 0\}$. They are domains defined by the inequalities of type $\{| \operatorname{Re} z | < C | \operatorname{Im} z |\}$. The sign of D^{\pm} is chosen so that the domain D^+ intersects the axis $\{\operatorname{Im} z > 0\}$.

Definition 2.6. A hyperbolic disc f has a *good (or admissible) approach* at $(0, 0)$ if

- (i) There exists a domain $D \subset \mathbb{C}$ which is asymptotic at 0 to one of the domains D^\pm (that is the boundary ∂D near $0 \in \partial D$ is formed by two curves tangent at the origin to the boundary lines of D^\pm).
- (ii) There exists a function g holomorphic on D such that in a neighborhood of the origin the image $f(\mathbb{D})$ is the graph of g over D .
- (iii) $g \in C^2(\overline{D})$ and $g(0) = g'(0) = g''(0) = 0$.

The first important consequence of Lemma 2.5 is the following assertion:

Lemma 2.7. *A hyperbolic disc has a good approach at a good hyperbolic point.*

Proof. As above, the function ψ in the left-hand side of (2.2) has the form $\psi(z) = r^2\chi(\theta)$ with $\chi(\theta) = 1 + \gamma \cos(2\theta)$. One readily sees that there is an angle $0 < \mu < \pi/4$ such that $\{\chi = 0\} = \{\theta = \pi/2 \pm \mu\}$. Let us write the two first terms of the expansion (2.3):

$$(2.5) \quad g(z) = az^{k/m} + bz^{k'/m} + o(z^{k'/m}), 2m < k < k'.$$

Consider a point $z = re^{i\theta(r)} \in \partial D$. It follows from (2.5) and from the identity $g(z) = r^2\chi(\theta(r))$, $z \in \partial D$, that $\chi(\theta(r)) = O(r^{k/m-2})$. Since the right-hand side converges to 0 as $r \rightarrow 0$, we conclude that $\theta(r)$ converges to $\pi/2 - \mu$ or to $\pi/2 + \mu$ as $r \rightarrow 0$ when z belongs to γ_+ or γ_- respectively. Thus, ∂D is tangent to the lines $\{\theta = \pi/2 \pm \mu\}$ at the origin. Q.E.D.

Another important consequence is the uniqueness principle for hyperbolic discs.

Lemma 2.8. *Suppose that the origin is a good hyperbolic point. Let f_1 and f_2 be hyperbolic discs in a neighborhood U of the origin which are the graphs of functions g_j over domains D_j . Suppose that one of the following conditions holds :*

- (i) $D_1 = D_2$,
- (ii) $D_1 \subset D_2$ and $\operatorname{Re} g_1(z) \leq \operatorname{Re} g_2(z)$, $z \in U \cap D_1$.

Then $f_1 = f_2$.

Proof. (i) This is immediate. Indeed, since $f_j(\partial\mathbb{D}) \subset S^2$, we have $g_j(z) = \psi(z)$ for $z \in \partial D$ near the origin i.e. $g_1(z) = g_2(z)$ on an arc of ∂D of positive length. Then $g_1 \equiv g_2$ by the classical boundary uniqueness theorem for holomorphic functions.

(ii) First we study more precisely the asymptotic behaviour of D at the origin. We use the equality $\{\theta = \pi/2 \pm \mu\} = \{Im az^{k/m} = 0\}$ established above. Let $a = r_0 e^{i\theta_0}$. Then the set $\{Im az^{k/m} = 0\}$ consists of the rays $R_l := \{\theta = (m/k)l\pi - (m/k)\theta_0\}$, l being an integer. For some l the rays R_l and R_{l+1} coincide with $\theta = \pi/2 \pm \mu$ which implies that $\pi m/k = 2\mu$ and $\theta_0 = \pi l + 2\pi - 2(k/m)\pi$. Set $\mu = -\chi'(\pi/2 - \mu) = \chi(\pi/2 + \mu)$. We have $\theta(r) = \pi/2 \pm \mu + o(r^{k/m-2})$ when $z \rightarrow 0$ along γ_\pm . Expanding the identity $\theta(r) = \chi^{-1}(r^{-2}g(z))$ at the origin we obtain

$$(2.6) \quad \theta(r) = \pi/2 \pm \mu - \frac{|a|}{\nu} r^{k/m-2} + O(r^{(k+1)/m-2}).$$

It is worth noticing here that the constants μ , k/m , ν are determined by γ and are independent of g .

Now we proceed the second step of the proof. Consider the Puiseux expansion $g_j(z) = a_j z^{k/m} + \dots$ for $j = 1, 2$. The boundary of every domain D_j is given by (2.6) near the origin. Since D_1 is contained in D_2 we obtain that $|a_1| = |a_2|$. For the same reason the rays R_l and R_{l+1} , corresponding to $j = 1, 2$, are defined by the same l . Hence $a_1 = a_2$. Next, consider

the difference

$$g_2(z) - g_1(z) = \sum_{n>k} c_n z^{n/m}$$

Let c_q be the first non-vanishing coefficient. Then

$$0 < \operatorname{Re} g_2(z) - \operatorname{Re} g_1(z) = \operatorname{Re}(c_q z^{q/m}) + o(z^{q/m}).$$

on D_1 near the origin. But $\operatorname{Re}(c_q z^{q/m})$ can be positive only on a sector of angle $\pi m/q$ while ∂D_1 is asymptotic to a sector of larger angle $\pi k/m$: a contradiction. Q.E.D.

3. INDICES

In the first part of our work [10] we saw that an extension of a 1-parameter family of Bishop discs is determined by some topological characteristic of these discs called in [10] the winding number. In the present work we need to study the behaviour of this invariant when a family of discs extends past a hyperbolic point. For our applications it is convenient to discuss this notion more conceptually and in full generality. This is the goal of the present section.

We recall some known facts concerning the Maslov index and the topological properties of real surfaces in (almost) complex manifolds.

3.1. Totally real case : the Maslov index. There are several possibilities to introduce V.Arnold's conception of the Maslov index [1]; we follow [16, 12]. Denote by S^1 the unit circle. Let

$$R(n) = GL(n, \mathbb{C})/GL(n, \mathbb{R})$$

be the manifold of totally real n -dimensional subspaces of \mathbb{C}^n . Consider the map $\kappa : R(n) \rightarrow S^1$ defined by

$$\kappa(B \cdot GL(n, \mathbb{R})) = \frac{\det(B^2)}{\det(B^*B)}, B \in GL(n, \mathbb{C})$$

where the index star denotes the matrix transposition and the complex conjugation. Let $\gamma : S^1 \rightarrow R(n)$ be a continuous map i.e. a loop in $R(n)$. The Maslov index of γ is defined by

$$\mu(\gamma) = \deg(\kappa \circ \gamma)$$

where \deg denotes the topological degree of a map. V.Arnold [1] proved that two loops in $R(n)$ are homotopic if and only if they have the same Maslov index. According to classical results a complex vector bundle L over the unit circle S^1 is trivial because $\pi_0(GL(n, \mathbb{C})) = 0$. If F is a totally real subbundle (with fibers of real dimension n) of the trivial bundle $L = S^1 \times \mathbb{C}^n$, we consider the loop $\gamma : S^1 \rightarrow R(n)$ defined by

$$(3.1) \quad \gamma(\zeta) = F_\zeta, \zeta \in S^1.$$

Here F_ζ denotes the fiber of F at the point $\gamma(\zeta)$. Hence it is a totally real subspace of \mathbb{C}^n and can be viewed as an element of $R(n)$. Since $\pi_0(GL(n, \mathbb{R})) = \mathbf{Z}_2$, there are two rank n real vector bundles over the circle. Two totally real subbundles are isomorphic as real bundles if their Maslov classes have the same parity.

Let $L \rightarrow \mathbb{D}$ be a complex rank n vector bundle over the unit disc \mathbb{D} and $F \subset L|_{\partial \mathbb{D}}$ be a totally real subbundle over $\partial \mathbb{D}$. The Maslov index $\mu(L, F)$ of the pair (L, F) is an integer which can be defined axiomatically by the properties of isomorphism, direct sum, normalisation and decomposition. Let us describe them.

Isomorphism. If $\Phi : L_1 \rightarrow L_2$ is a vector bundle isomorphism then

$$\mu(L_1, F_1) = \mu(L_2, \Phi(F_1)).$$

Direct sum.

$$\mu(L_1 \oplus L_2, F_1 \oplus F_2) = \mu(L_1, F_1) + \mu(L_2, F_2).$$

Normalisation. If $L = \mathbb{D} \times \mathbb{C}$ is the trivial line bundle and $F_\zeta = e^{ik\theta/2}\mathbb{R}$, $\zeta = e^{i\theta} \in \partial\mathbb{D}$, then

$$\mu(L, F) = k.$$

We do not describe the decomposition property (see [16]) since we do not need it here.

In general one can show that if $L = \mathbb{D} \times \mathbb{C}^n$ is the trivial bundle, then

$$\mu(L, F) = \mu(\gamma)$$

where the loop γ in $R(n)$ is defined by (3.1). One can use this property as the definition of the Maslov class and then show (see [16]) that it is independent of a choice of trivialisation. The following trivialisation presented in [16] is particularly useful.

Lemma 3.1. *For every complex line bundle L over \mathbb{D} and every totally real subbundle $F \subset L|_{\partial\mathbb{D}}$ there exists a trivialisation such that in the corresponding coordinates one has $F_\zeta = e^{ik\theta/2}$, $\zeta = e^{i\theta} \in \partial\mathbb{D}$.*

Let now f be a Bishop disc with boundary attached to a totally real manifold E in an almost complex manifold M . The pull-back $f^*(TM)$ is a complex vector bundle over \mathbb{D} and $f^*(TE)$ is its totally real subbundle over $\partial\mathbb{D}$. Then the Maslov class $\mu(f^*TM, f^*TE)$ is defined and is called *the Maslov index of the disc f* . We denote it by $\mu_E(f)$.

It was shown in [10] that every elliptic point of the sphere S^2 generates a 1-parameter family of Bishop discs (f^t) whose boundaries foliate a punctured neighborhood of this point in S^2 . We introduced in [10] a topological invariant of f^t called the winding number. It is equal to 0 for every disc f^t . Comparing that definition of the winding number for a Bishop disc f^t from [10] with Lemma 3.1 one readily sees that the Maslov index of f^t also is equal to 0. Thus, the Maslov index of every disc f^t generated by an elliptic point is equal to 0. We sum up this in the following assertion.

Lemma 3.2. *Let (f^t) be a 1-parameter family of Bishop discs generated by an elliptic point as in [10]. Then the winding number of each disc coincides with its Maslov index. In particular*

$$\mu_{S^2}(f^t) = 0.$$

3.2. Index of a complex point. Here we follow [8, 17]. Let S be a real surface embedded into an almost complex manifold M of complex dimension 2. Recall that we assume everywhere that the complex points of S are isolated and are either elliptic or hyperbolic. Assume for simplicity that the almost complex structure J is integrable near the complex points of S^2 ; this assumption can be easily dropped. Let p be a complex point in S . There exist local complex coordinates (z, w) centered at p such that S is locally the graph $\{(z, w) : w = g(z)\}$ of a smooth complex valued function g defined in a neighborhood of the origin in \mathbb{C} and such that $g(z) = O(|z|^2)$. The points where $\partial g / \partial \bar{z} \neq 0$ are totally real. Since the origin is an isolated complex point, $\partial g / \partial \bar{z}$ does not vanish elsewhere in a neighborhood of the origin. The index of p , denoted by $I(p, S)$ is defined as the winding number of the function $\partial g / \partial \bar{z}$ around the origin. In the general almost complex situation consider first the case of an elliptic point. Choose local coordinates as in [10]; in particular, $J(0) = J_{st}$. Then the origin

is also an isolated complex point for the standard structure and we can apply the above construction using the standard operator $\partial g/\partial \bar{z}$. Since we consider here the case where an almost complex structure is integrable near each hyperbolic point, the above construction can be applied directly there. We point out that $I(p, S)$ does not depend on a choice of g defining S^2 locally. For a totally real point $p \in S$ we set $I(p, S) = 0$. One readily sees from this definition that the index of an elliptic point is equal to $+1$ and the index of a hyperbolic point is equal to -1 . This definition does not rely on an orientation of S .

If S is orientable (which is the case considered in the present paper) the orientation must be taken into account.

Definition 3.3. A complex point p of S^2 is called *positive* (resp. *negative*) if the orientation of the tangent space $T_p S^2$ coincides with (resp. is opposite to) the orientation induced by (M, J) .

We define $I_+(S)$ (resp. $I_-(S)$) as the sum of the indices over positive (resp. negative) complex points in S . It is known that $I_+(S) = I_-(S)$ for every closed oriented immersed surface in \mathbb{C}^2 , see [8].

The index of S is defined by $I(S) := \sum_{p \in S} I(p, S)$; in particular $I(S) = I_+(S) + I_-(S)$. We note that for a sphere S generically embedded into a (almost) complex manifold we have

$$I(S) = \chi(S) = 2$$

where $\chi(S)$ is the Euler number.

We denote by $e_+(S)$ (resp. $e_-(S)$) the number of positive (resp. negative) elliptic points of S and by $h_+(S)$ (resp. $h_-(S)$) the number of positive (resp. negative) hyperbolic points. Then from this definition we get $I_{\pm}(S) = e_{\pm}(S) - h_{\pm}(S)$. Moreover, if S is an oriented real surface embedded to a complex surface M we have :

Lemma 3.4.

$$(3.2) \quad I_{\pm}(S) = (1/2)(I(S) \pm c(S)).$$

Here $c(S)$ denotes the value of the first Chern class $c_1(M)$ on S . For example, $c(S)$ always vanishes when $M = \mathbb{C}^2$; in what follows we assume that this condition always holds in the present paper. In particular we have : $I_{\pm}(S^2) = 1$ for a two-sphere S^2 embedded into an almost complex manifold.

In [8] a slightly different version of the notion of the index of a loop is used. Let us recall it since this is useful in index computations. The simplest way is to define it in a special system of coordinates. Let U be an open set in the complex plane, $g : U \rightarrow \mathbb{C}$ be a smooth complex valued function on U , and $E \subset \mathbb{C}^2$ be the graph of g :

$$(3.3) \quad E = \{(z, g(z)) \in \mathbb{C}^2 : z \in U\}.$$

Let now $\gamma = (\gamma_1, \gamma_2) : S^1 \rightarrow E$ be a loop contained in the set of totally real points of E . Then the index $I_E(\gamma)$ is equal to the winding number of the function $\theta \in S^1 \mapsto \partial g/\partial \bar{z}(\gamma_1(\theta)) \in \mathbb{C} \setminus \{0\}$. In some situations it is not convenient to use a coordinate representation (3.3). If E is embedded to \mathbb{C}^2 , the simplest way to compute the index $I_E(\gamma)$ is the following (see [8]). Choose continuous vector fields $X_j : S^1 \rightarrow \mathbb{C}^2$, $j = 1, 2$ such that for every $\theta \in S^1$ the vectors $X_j(\theta)$ form a basis of $T_{\gamma(\theta)} E$. Such vector fields exist when E is orientable along γ which is always our case; the method can be extended to the non-orientable case, see [8]. Then the index $I_E(\gamma)$ is equal to the winding number (around the origin) of the

determinant $\det(X_1(\theta), X_2(\theta))$. This determinant does not vanish on the unit circle since E is totally real. Hence $I_E(\gamma)$ is correctly defined and is independent on a choice of X_j , in particular, is independent on an orientation of E (but depends, of course, on an orientation of γ). Indeed, if X'_j , $j = 1, 2$ is another couple of vector fields with similar properties, then $X'_j = AX_j$ for some continuous map $A : S^1 \rightarrow GL(2, \mathbb{R})$. Hence the winding number of $\det(X'_1, X'_2)$ is equal to the sum of winding numbers of $\det A$ and $\det(X_1, X_2)$. But $\det A$ is a real-valued non-vanishing function on S^1 , so its winding number is equal to 0. One can give an intrinsic definition of $I_E(\gamma)$ for an oriented real surface embedded or immersed into a (almost) complex manifold. This general definition is similar to the notion of the Maslov index so we drop it, see details in [8].

We recall here that the Maslov index of the boundary $f|_{\partial\mathbb{D}}$ of a Bishop disc generated by an elliptic point is equal to 0. On the other hand, a direct computation shows that $I_{S^2}(f|_{\partial\mathbb{D}}) = 1$. It is easy to see from the above definitions that for a Bishop disc f the equality $\mu_{S^2}(f) = 0$ holds if and only if $I_{S^2}(f|_{\partial\mathbb{D}}) = 1$. Indeed, choose canonical coordinates along $f(\mathbb{D})$ as in [10]; they provide a trivialisation of bundles from Lemma 3.1. Comparing the Maslov index and the winding number of the determinant as described above (with an obvious choice of vector fields X_j), we conclude.

The following useful statement is contained in [8].

Proposition 3.5. *Let E be an oriented real surface with totally real boundary ∂E . Then*

$$I_+(E) - I_-(E) = I_E(\partial E).$$

The above mentioned results are often proved for real surfaces in complex manifolds. It is easy to see that the proofs remain true without any changes in the almost complex case since they use only standard differential geometry and the topological properties of complex vector bundles.

4. APPROACHING GOOD HYPERBOLIC POINTS BY FAMILIES OF DISCS

Let f^t , $t \in]0, 1[$, be a one-parameter family of embedded J -holomorphic discs of Maslov index 0 attached to the totally real part S_*^2 of the sphere S^2 . They have a uniformly bounded area and their boundaries $f^t(\partial\mathbb{D})$ foliate an open subset E of S_*^2 , see [10]. Let $(f_k)_k$ be a sequence of such discs corresponding to the values (t_k) of the parameter i.e. $f_k = f^{t_k}$. Since the areas are bounded, Gromov's compactness theorem can be applied. The case where these discs are separated from the set of complex points of S^2 is considered in [10]. In that case there are no bubbles and after a suitable reparametrization by Mobius transformations the sequence (f_k) converges in every $C^m(\overline{\mathbb{D}})$ norm to a non-constant Bishop disc attached to S_*^2 . Let us consider now the case where the limit of a sequence of discs touches a good hyperbolic point p . The usual version of Gromov's compactness theorem deals with totally real manifolds. Here we adapt it to our situation. Previously we often identified a map $f_k : \mathbb{D} \rightarrow M$ and its image $f_k(\mathbb{D})$ using the same terminology "disc" for both of them. In this section we proceed more carefully and distinguish the convergence of $(f_k(\mathbb{D}))_k$ as a sequence of sets (i.e. non-parametrized holomorphic curves) and the convergence of $(f_k)_k$ as a sequence of maps (i.e. parametrized holomorphic curves).

4.1. Convergence of (f_k) as sets. Let U be a neighborhood of a good hyperbolic point p . Then $Y_k = f_k(\mathbb{D}) \cap U$ is a closed complex purely 1-dimensional analytic subset in $U \setminus S^2$ for every integer $k \geq 1$. Since the areas of the sets Y_k are uniformly bounded, Bishop's

convergence theorem [4] implies (after extracting a subsequence) that the sequence (Y_k) converges on every compact subset of $U \setminus S^2$ to a complex purely 1-dimensional analytic subset Y in $U \setminus S^2$. Here the convergence is in the sense of the Hausdorff distance. Applying Lemma 2.3 to Y we conclude that Y extends to a neighborhood of p as a complex 1-dimensional analytic set. Thus, the convergence of $(f_k(\mathbb{D}))$ as sets near a hyperbolic point is quite simple. Unfortunately, this argument is not precise enough. In order to obtain a more detailed information about the limit analytic set Y , we need to study the convergence of the sequence of maps (f_k) .

4.2. Interior convergence of (f_k) as maps. We proceed in several steps. First, recall some basic notions related to the Gromov compactness theorem that we apply to our sequence $(f_k)_k$ (see [10] and references there for precise definitions and statements). After a suitable reparametrization we may assume that (f_k) converges to a non-constant map f_∞ attached to S^2 . The convergence is in *Gromov's sense* which we briefly discuss.

We say that a *spherical bubble* arises at an interior point q of the unit disc \mathbb{D} if one can find a sequence of biholomorphic maps (ϕ_k) which "blow up" a neighborhood of q such that the sequence $(f_k \circ \phi_k)$ converges to a non-constant J -holomorphic map $g : \mathbb{C} \rightarrow M$. Since the areas are bounded, g extends to the whole Riemann sphere as a J -holomorphic map and is called a spherical bubble. Similarly we may define *disc-bubbles* that can occur only at boundary points of \mathbb{D} . Since the areas of f_k are uniformly bounded with respect to k , bubbles can occur only in a finite number of points $\Sigma \subset \overline{\mathbb{D}}$. The sequence (f_k) converges to the limit disc f_∞ uniformly on every compact subset of $\overline{\mathbb{D}} \setminus \Sigma$ and at every point of the set Σ a bubble necessarily arises. After a suitable reparametrization by Möbius transformations the sequence converges to a prescribed bubble. The images $f_k(\overline{\mathbb{D}})$ converge in the Hausdorff distance to the union of $f_\infty(\mathbb{D})$ and of the images of the bubbles. Furthermore, this union is a *nodal curve*, in particular, is connected. Recall that *the standard node* is the complex analytic set $\{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\}$. A point on a complex curve is called a *nodal point* if it has a neighborhood biholomorphic to the standard node. A boundary nodal point may be defined using the double of a given complex curve with boundary (namely its compactification as a Riemann surface, see, for instance, [18]). A nodal curve is a compact complex curve with boundary and with a finite number of interior and boundary nodes (see [13, 18] for more details). Notice that in the case under consideration the following simplification occurs. According to Section 2, every hyperbolic disc extends as a complex analytic set through a good hyperbolic point. Hence, if a disc-bubble arises at a good hyperbolic point p , this point p will be an interior nodal point for a complex analytic set in a neighborhood of p which extends "the principal disc" and a bubble.

In what follows we need a more detailed description of the behaviour of the sequence (f_k) near a point where a disc bubble arises. We follow [18, 19, 21]. Let q be a point in $\Sigma \in \partial\mathbb{D}$. Then one can find a sequence of discs $D_k := q_k + r_k\mathbb{D}$ of radius $r_k \rightarrow 0$, centered at $q_k \in \partial\mathbb{D} \rightarrow q$, and conformal injective maps $\psi_k : D_k \cap \mathbb{D} \rightarrow H := \{\zeta \in \mathbb{C} : \text{Im } \zeta > 0\}$ such that $\psi_k(D_k \cap \partial\mathbb{D}) \subset \partial H$, $\psi_k(D_k \cap \mathbb{D}) \rightarrow H$ and $f_k \circ \psi_k^{-1}$ converges smoothly on compacts to a disc-bubble g (more precisely, g becomes a disc bubble after a composition with a conformal isomorphism from \mathbb{D} to H). Moreover, $f_k(\partial D_k \cap \mathbb{D})$ converges to a cusp (nodal) point $f_\infty(q)$ where f_∞ and g intersect each other.

The connectedness of the limit nodal curve is often useful to study its global topological properties (the Maslov indices of discs contained in such a curve, their homology classes,

etc.). Furthermore, for large k the homology class $[f_k|_{\partial\mathbb{D}}]$ is equal to the sum of $[f_\infty|_{\partial\mathbb{D}}]$ and of the homological classes of bubbles boundaries on S^2 (see, for instance, [18, 21]).

Recall that the first step in the proof of Gromov's compactness theorem [16, 19] considers holomorphic discs with free boundary (i.e. without totally real boundary conditions) and claims that after a suitable reparametrization the sequence (f_k) converges to a non-constant disc f_∞ outside a finite set Σ as explained above. By assumption, M contains no non-constant J -holomorphic spheres. Since spherical bubbles are non-constant, there are no spherical bubbles. Hence the only possibility is that the set Σ where bubbles arise is contained in $\partial\mathbb{D}$ and the bubbles must be disc-bubbles. In particular, the sequence (f_k) converges to f_∞ uniformly on compact subsets of \mathbb{D} . But then, since the areas of the discs are uniformly bounded, the sequence (f_k) satisfies the assumptions of the theorem in [15] which implies that the cluster set $C(f_\infty, \partial\mathbb{D}) = \overline{f_\infty(\mathbb{D})} \setminus f_\infty(\mathbb{D})$ is contained in S^2 . The same holds for the boundary disc-bubbles since after suitable reparametrization the sequence (f_k) converges uniformly on compact subsets of \mathbb{D} to such a bubble.

Our next goal is to study the behaviour of boundary disc bubbles. We will see that they do not arise at totally real boundary points of a hyperbolic disc and will give a precise description of bubbling near a good hyperbolic point. We begin with a useful technical statement mentioned above in Section 2.

4.3. Boundary continuity of discs forming the limit nodal curve. Let $f : \mathbb{D} \rightarrow M$ be a non-constant J -holomorphic disc from the limit nodal curve (i.e. after a suitable reparametrization by a sequence of Möbius transformations the sequence (f_k) converges to f). As we noticed above, the cluster set $C(f, \partial\mathbb{D}) = \overline{f(\mathbb{D})} \setminus f(\mathbb{D})$ of f on $\partial\mathbb{D}$ is contained in S^2 . If S^2 would be a totally real manifold, Gromov's compactness theorem would imply that f is smooth on $\overline{\mathbb{D}}$. However, the presence of hyperbolic points requires additional considerations. The following assertion is due to [11].

Proposition 4.1. *The map f extends continuously on $\overline{\mathbb{D}}$.*

Proof. Since the area of f is bounded, for every point $p \in S^2$ and $\varepsilon > 0$ small enough the intersection $f(\mathbb{D}) \cap B(p, \varepsilon)$ admits a finite number of connected disc-components in view of Lemma 2.4. We notice here that if p is a good hyperbolic point, then it follows from the description of the boundary behaviour of f near such a point established in Section 2 that the area of every disc in $f(\mathbb{D}) \cap B(p, \varepsilon)$ is separated from zero.

Let $p \in S_*^2$ be a totally real point of S^2 . Lemma 2.4 may be applied to every map f_k on $B(p, \varepsilon)$. Since the areas of f_k are bounded from above, the number of discs in $f_k(\mathbb{D}) \cap B(p, \varepsilon)$ is bounded from above, independently of k . We use here the well-known fact (see for instance [16]) that the areas of these discs are separated from zero. Applying to these discs Gromov's compactness theorem, we obtain that their boundaries converge in $B(p, \varepsilon) \cap \partial M$ to a continuous curve which is a connected finite union of smooth curves (as we will prove below, the bubbles do not arise here so in fact the limit will be a single smooth curve). Thus, the boundary of $f(\mathbb{D})$ near a totally real point p is a continuous curve. If p is a hyperbolic point, the boundary behavior of f is described in Section 2; as it is proved there (see the basic representation (2.3)), the boundary of $f(\mathbb{D})$ near such a point also is a continuous curve.

Now we may proceed the proof of the continuity of f . This is a slight modification of the classical Geometric Function Theory argument.

Suppose by contradiction that f does not extend continuously to a point $\zeta_0 \in \partial\mathbb{D}$. Then there exist two sequences (ζ_n) , $(\tilde{\zeta}_n)$ converging to ζ_0 such that $p_n = f(\zeta_n)$ and $\tilde{p}_n = f(\tilde{\zeta}_n)$ converge respectively to $p_\infty \neq \tilde{p}_\infty$. Since the almost complex structure J is tamed by the symplectic form ω , they define canonically a Riemannian metric g (see, for instance, [16]):

$$g(u, v) = (1/2)(\omega(u, Jv) + \omega(v, Ju))$$

We will measure the distances and norms with respect to this metric (in fact, any Riemannian metric satisfying $g(X, X) \leq \omega(X, JX)$ is adapted). Let $d = \text{dist}(p_\infty, \tilde{p}_\infty)$. We may assume that $d(p_n, p_\infty) \leq d/3$, $d(\tilde{p}_n, \tilde{p}_\infty) \leq d/3$ for all n . Denote by $B(p, \varepsilon)$ the ball of radius ε centered at p .

In view of the above description of the boundary behaviour of $f(\mathbb{D})$ near $C(f, \partial\mathbb{D})$, there exist maps (paths) λ and $\tilde{\lambda}$, continuous on $[0, 1]$, such that $\lambda([0, 1)) \subset f(\mathbb{D}) \cap B(p_\infty, d/3)$, $\tilde{\lambda}([0, 1)) \subset f(\mathbb{D}) \cap B(\tilde{p}_\infty, d/3)$, and $\lambda(1) = p_\infty$, $\tilde{\lambda}(1) = \tilde{p}_\infty$. Furthermore, there exist increasing sequences $t_n, \tilde{t}_n \rightarrow 1$ such that $\lambda(t_n) = p_n$, $\tilde{\lambda}(\tilde{t}_n) = \tilde{p}_n$ (passing to a subsequence if necessarily).

Set $\Lambda(t) = f^{-1}(\lambda(t))$, $\tilde{\Lambda}(t) = f^{-1}(\tilde{\lambda}(t))$. Let $t_1(r)$ denote the smallest $t \in [0, 1]$ such that $\Lambda(t) \in \partial B(\zeta_0, r)$. The function $r \mapsto t_1(r)$ is decreasing, so is continuous except at most on a countable set of points of discontinuity. Put $\Lambda(t_1(r)) = \zeta(r) = \zeta_0 + re^{i\tau(r)}$. Then the functions $r \mapsto \zeta(r)$ and $r \mapsto \tau(r)$ also are continuous except on a countable set. We may define similarly the functions $\tilde{\zeta}$ and $\tilde{\tau}$.

Fix $r_0 > 0$ small enough. Using the polar coordinates $\zeta = re^{i\theta}$ on the disc $\zeta_0 + r_0\mathbb{D}$ and integrating along the arc $[\tau(r), \tilde{\tau}(r)]$ of $\zeta_0 + r_0\partial\mathbb{D}$, we obtain :

$$d/3 \leq d(f(p_n), f(\tilde{p}_n)) \leq \int_{[\tau(r), \tilde{\tau}(r)]} \left\| Df \left(\frac{1}{r} \frac{\partial}{\partial \theta} \right) \right\| r d\theta.$$

By the Cauchy-Schwarz inequality we get :

$$\begin{aligned} (d/3)^2 &\leq r^2 |\tau(r) - \tilde{\tau}(r)| \int_{[\tau(r), \tilde{\tau}(r)]} \left\| Df \left(\frac{1}{r} \frac{\partial}{\partial \theta} \right) \right\|^2 d\theta \\ &\leq 2\pi r^2 \int_{[\tau(r), \tilde{\tau}(r)]} \left\| Df \left(\frac{1}{r} \frac{\partial}{\partial \theta} \right) \right\|^2 d\theta. \end{aligned}$$

Let $dm(\zeta)$ denote the standard Lebesgue measure on \mathbb{C} with respect to the variable ζ . Dividing by r and integrating with respect to r from ε to r_0 we obtain :

$$\begin{aligned} (d/3)^2 \ln(r_0/\varepsilon) &\leq 2\pi \int_{[\varepsilon, r_0]} r dr \int_{[\tau(r), \tilde{\tau}(r)]} \left\| Df \left(\frac{1}{r} \frac{\partial}{\partial \theta} \right) \right\|^2 d\theta \\ &\leq 2\pi \int_{\mathbb{D} \cap (\zeta_0 + r_0\mathbb{D})} \left\| Df \left(\frac{1}{r} \frac{\partial}{\partial \theta} \right) \right\|^2 dm(\zeta) \\ &\leq 2\pi \int_{\mathbb{D}} \omega \left(Df \left(\frac{1}{r} \frac{\partial}{\partial \theta} \right), Df \left(i \frac{1}{r} \frac{\partial}{\partial \theta} \right) \right) dm(\zeta) \\ &= 2\pi \text{area}(f). \end{aligned}$$

Here in the last inequality we used the classical identity connecting the symplectic area of a J -holomorphic curve and the g -norm of its tangent vectors, see [16], pp. 20-21. Since $\varepsilon \rightarrow 0$, we obtain a contradiction which proves the Proposition. Q.E.D.

4.4. Disc-bubbles do not arise at totally real points. The next step is the following

Lemma 4.2. *If $p \in f(\partial\mathbb{D})$ is a totally real point of S^2 , there are no disc bubbles arising at p .*

Proof. We could proceed similarly to [10], but in the presence of hyperbolic points it requires some global analysis of the characteristic foliation. So we use another argument due to [21], easy to localize. Suppose that a disc-bubble g arises at a totally real point $p = f_\infty(\zeta_1)$, $\zeta_1 \in \partial\mathbb{D}$. Let $\zeta_2 \in \partial\mathbb{D}$ be a point with $g(\zeta_2) = p$. By Proposition 2.6 of [10] the maps $f_\infty|_{\partial\mathbb{D}}$ and $g|_{\partial\mathbb{D}}$ are embeddings near ζ_1 and ζ_2 respectively. Then we may fix open arcs $\gamma_j \subset \partial\mathbb{D}$ containing ζ_j , $j = 1, 2$, such that $g(\gamma_2)$ lies on one side of $f_\infty(\gamma_1)$; in particular, $f_\infty(\gamma_1)$ and $g(\gamma_2)$ are tangent at p (if not, the curves $f_k(\partial\mathbb{D})$ would have self-intersections for large k , but they are embedded). Furthermore, because of the above mentioned relation for homological classes, the circles $f_\infty(\partial\mathbb{D})$ and $g(\partial\mathbb{D})$ have opposite orientations. Let τ be a continuous unit tangent vector on the unit circle $\partial\mathbb{D}$. Set $Y_k = \frac{\partial f_k}{\partial \tau} \left\| \frac{\partial f_k}{\partial \tau} \right\|^{-1}$. Then, see [21], it follows from the above description of a disc-bubble that there exists sequences (ξ_k) and $(\tilde{\xi}_k)$ on γ_1 , converging to ζ_1 and such that $f_k(\xi_k) \rightarrow p$, $f_k(\tilde{\xi}_k) \rightarrow p$ and

$$\lim_{k \rightarrow \infty} Y_k(\xi_k) = v, \lim_{k \rightarrow \infty} Y_k(\tilde{\xi}_k) = -v$$

where v is a unit tangent vector to $f_\infty(\partial\mathbb{D})$ at p . Let now χ be the characteristic foliation on S^2 , see [10]. Choose a continuous unit tangent vector field X on S^2_* which is everywhere orthogonal to χ (with respect to the inner product \bullet induced by some Riemannian metric). By Proposition 2.6 of [10] f_∞ is transverse to χ at p ; hence $v \bullet X(p) \neq 0$. Then for large k the products $\frac{\partial f_k}{\partial \tau}(\xi_k) \bullet X(f_k(\xi_k))$ and $\frac{\partial f_k}{\partial \tau}(\tilde{\xi}_k) \bullet X(f_k(\tilde{\xi}_k))$ have opposite signs. By the intermediate value theorem, there exists a point $\eta_k \in \gamma_1$ such that $\frac{\partial f_k}{\partial \tau}(\eta_k) \bullet X(f_k(\eta_k)) = 0$. This means that f_k is tangent to χ at the point $f_k(\eta_k)$, which contradicts Proposition 2.6 of [10]. Q.E.D.

In particular, it follows by Gromov's compactness theorem that the sequence (f_k) converges to f up to the boundary in each C^k -norm near every totally real point $p \in S^2_*$. Furthermore, since ∂M is J -convex, it follows by [10] that f is transverse to ∂M at p .

Now we study the behaviour of (f_k) near a hyperbolic point.

4.5. Dynamics of (f^t) near a good hyperbolic point. Let f^t , $t \in \mathbb{R}$, be a 1-parameter family of embedded Bishop discs attached to the totally real part S^2_* of the sphere S^2 and converging to a non-constant hyperbolic disc f^∞ . We call such a family *maximal*. Let p be a good hyperbolic point in the boundary of f^∞ . We suppose that local coordinates near p are given by Definition 2.2. Therefore it follows from Section 2 that every disc f^t is the graph of a function g^t holomorphic in a domain D^t in \mathbb{C} , in a neighborhood of the origin. The boundaries of the discs f^t are disjoint so we may assume that the family D^t of domains in \mathbb{C} is either increasing or decreasing.

Definition 4.3. The family f^t *approaches f^∞ from inside* at p (resp. *from outside*) if the family D^t is increasing (resp. decreasing).

In order to determine which case actually occurs near a given hyperbolic point, it is convenient to use the orientability of S^2 . As above, consider a 1-parameter family (f^t) ,

$t \in \mathbb{R}$, of Bishop discs attached to the totally real part S_*^2 . We observe that such a family (f^t) provides S^2 with an orientation. This orientation is defined by pushing forward the form $dt \wedge d\theta$ via the map

$$(t, \theta) \mapsto f^t(e^{i\theta}).$$

Definition 4.4. The family f^t is called *positive* if this orientation coincides with an orientation already fixed on S^2 . Otherwise a maximal family is called *negative*.

We have the following

Lemma 4.5. *A positive elliptic point generates a positive family of Bishop discs.*

Proof. Consider first the model case where S^2 is defined near an elliptic point by the equation (2.2) with $0 < \gamma < 1$. The Bishop discs near the origin are described in [10] and are obtained by the intersection with the hyperplanes $\{Re w = t\}$. One readily sees that in this case the statement of lemma 4.5 holds. Since the general case is a small perturbation of this model situation (see [10]) the assertion remains true. Q.E.D.

Lemma 4.6. *Suppose that the family of discs $(f^t(\overline{\mathbb{D}}))_k$ approaches a good hyperbolic point p from inside. Then (after a suitable reparametrization by Mobius transformations) the family (f^t) converges (passing to a subsequence) in $C(\overline{\mathbb{D}})$ to a hyperbolic disc.*

Proof. We use local coordinates (z, w) near p as in Section 2, identifying p with the origin. Then as $t \rightarrow \infty$ every disc $f^t(\mathbb{D})$ near 0 is the graph $\{w = g^t(z)\}$ of a function g^t holomorphic over a domain D^t with smooth boundary. Let $t_k \rightarrow \infty$ be an increasing sequence of values of parameter, $f_k = f^{t_k}$. The family (D^t) is monotone and converges to a domain D with ∂D smooth off the origin D ; this domain D satisfies Definition 2.6 of Section 2. Since the family (D^t) is increasing, the limit domain D can fill only one admissible region i.e. is asymptotic either to the domain D^+ or D^- from Definition 2.6.

Next, the family (g^{t_k}) converges to a function g holomorphic on D near the origin. According to Section 2, g is of class $C^2(\overline{D})$ and also satisfies Definition 2.6. As we saw previously, the intersection of the limit nodal curve with a neighborhood of the origin is a finite number of holomorphic discs in M which are graphs over D . But in our case the limit of discs $(f_k(\mathbb{D}))$ is the graph of g over D near the origin so only one single disc, the graph of g , appears in the limit. Hence the origin is not a nodal point and no disc bubble arises at the origin. If $t_n \rightarrow \infty$ is another sequence of parameters and $f_n = f^{t_n}$, we repeat the same argument. Since (D^t) is a monotone family, the domains (D^{t_n}) converge to the same domain D which is the limit of (D^{t_k}) . Suppose that the limit of $(f_n(\mathbb{D}))$ is the graph of \tilde{g} over D in a neighborhood U of the origin. Then $\tilde{g}(z) = g(z) = \psi(z)$ for $z \in U \cap \partial D$, where ψ comes from Equation (2.2). By the boundary uniqueness theorem for holomorphic functions, $\tilde{g} = g$ on D . This means that the family (f^t) converges to a single disc which is the graph of g and the absence of bubbles implies the convergence in $C(\overline{\mathbb{D}})$. As it was mentioned previously, near totally real points the convergence will be in every C^k norm up to the boundary. Q.E.D.

In the case of outside approach bubbles necessarily arise, but their structure is quite simple in view of a local description of hyperbolic discs from Section 2.

Lemma 4.7. *Suppose that the family of discs $(f^t(\overline{\mathbb{D}}))_k$ approaches a good hyperbolic point p from outside. Then near p it converges in the Hausdorff distance to the union of two hyperbolic discs approaching p from opposite sides in admissible regions described in Section 2. Thus, the family of maps (f^t) converges to one of these discs after a suitable reparametrization by Mobius transformations and the second disc may be viewed as a bubble.*

Proof. We use the characteristic foliation χ induced by ∂M on S^2 , see [10]. Since p is a good hyperbolic point, we may assume that S^2 has the form (2.2) near p and $p = 0$ in these coordinates. Consider the projection $\pi : (z, w) \mapsto z$. Then the images $\pi(\chi)$ foliate a neighborhood of the origin in \mathbb{C} . They are trajectories of a first order dynamical system

$$\begin{cases} \operatorname{Re} \dot{z} &= (2\gamma + 1)\operatorname{Re} z + \alpha_1(2\gamma - 1)\operatorname{Im} z + O(z^2) \\ \operatorname{Im} \dot{z} &= \alpha_1(2\gamma + 1)\operatorname{Re} z - (2\gamma - 1)\operatorname{Im} z + O(z^2) \end{cases}$$

with α_1, γ given by the expansion of the local defining function ρ in Subsection 2.1. Then the origin is a saddle point for this dynamical system. There are four trajectories through the origin tangent to two lines through the origin; these lines are determined by the eigenvalues of the linear part of the above system. They divide a neighborhood of the origin in four regions, say, Ω_j , $j = 1, 2, 3, 4$, which are filled by other leaves of χ precisely as in the classical phase portrait of a dynamical system near a saddle point. Now suppose by contradiction that (f^t) is a family approaching p from outside and converging to a single disc. Then the limit disc is the graph over an admissible approach region D , say asymptotic to the domain D^+ from Definition 2.6. The discs $f^t(\mathbb{D})$ are the graphs over domains D^t with smooth boundaries near the origin; these domains decrease to D . But then for t large enough their boundaries ∂D^t intersect at least three regions Ω_j and one can find one region, say Ω_1 , where the intersections $\partial D^t \cap \Omega_1$ form a sequence of curves closed in Ω_1 , converging to the origin. But then, since the origin is a saddle point, for a given t one can find a leaf of the above dynamical system which is tangent to ∂D^t at some point $a \in \Omega_1$. Therefore, there exists a leaf of the characteristic foliation χ tangent to $f^t(\overline{\mathbb{D}})$ at some point. However, it is shown in [10] that this is impossible. Q.E.D.

Next, we have the following useful

Lemma 4.8. *If a positive (resp. negative) family (f^t) approaches a positive hyperbolic point p , then it must approach from inside (resp. outside).*

Once again, for the proof it suffices to consider the case where S^2 is defined near a positive hyperbolic point by (2.2) with $\gamma > 1$. In view of Lemma 2.7 the dynamics of the family f^t near the origin is the same as the behaviour of the sections by the real hyperplanes $\{Rw = t\}$. We conclude the proof by checking the orientations of boundaries of discs corresponding to these sections. Q.E.D.

Thus for every maximal family (f^t) two cases may occur. In the first case (inside approach) the family $(f^t(\mathbb{D}))$ converges to the image of a single hyperbolic disc $f_1(\mathbb{D})$ having a good approach. Then necessarily we deal with the inside approach and the boundaries of f^t fill an approach region near a good hyperbolic point p . In the second case (outside approach) the sequence $(f^t(\mathbb{D}))$ of discs converges to the union of images of two hyperbolic discs $f_1(\mathbb{D})$ and $f_2(\mathbb{D})$.

5. DEFORMATION OF HYPERBOLIC DISCS

Recall again [10] that a Bishop disc with boundary glued to S_*^2 belongs to a 1-parameter family of Bishop discs with boundaries foliating an open piece of S_*^2 . We establish here an analog for hyperbolic discs.

5.1. Gluing two hyperbolic discs into a single disc. The main technical result here is the following

Proposition 5.1. *Let f_1 and f_2 be two distinct hyperbolic discs at a good hyperbolic point p . Then given $\varepsilon > 0$ there exists an almost complex structure J_ε and a sphere S_ε^2 with the following properties:*

- (a) *the structure J_ε is integrable near p and coincides with J outside a neighborhood of p ; the sphere S_ε^2 also coincides with S^2 outside a neighborhood of p ;*
- (b) *$J_\varepsilon \rightarrow J$ in the C^1 -norm and $S_\varepsilon^2 \rightarrow S^2$ in the C^2 -norm as $\varepsilon \rightarrow 0$;*
- (c) *there exists a J_ε -holomorphic disc f_ε , ε -close to $f_1(\mathbb{D}) \cup f_2(\mathbb{D})$ in the Hausdorff distance, coinciding with $f_1(\mathbb{D}) \cup f_2(\mathbb{D})$ outside a neighborhood of p and such that its boundary is glued to the totally real (with respect to J_ε) part of S_ε^2 . The family (f_ε) tends to $f_1(\mathbb{D}) \cup f_2(\mathbb{D})$ from outside as $\varepsilon \rightarrow 0$.*

Proof. Step 1: smooth gluing of discs. We assume that S^2 is given by (2.2) near the origin which is a good hyperbolic point. Fix $\varepsilon > 0$ small enough. According to Section 2, the discs f_j have a good approach at p and in particular, they approach p from opposite regions. According to Lemma 2.7, the discs f_j are the graphs of holomorphic (with respect to J_{st}) functions $w = g_j(z)$ over domains D_j in \mathbb{C} asymptotic to the origin (the domains D_j are asymptotic to the domains D^\pm by Definition 2.6). Furthermore, the expansions of g_1 and g_2 coincide at the origin up to the second order. In particular there exists a real number α , $0 < \alpha < 1$, and a $C^{2,\alpha}$ -smooth real surface Π in a neighborhood of the origin such that $f_j(\mathbb{D})$ are contained in Π . Then $\Pi = \{w = o(|z|^2)\}$. There exists a $C^{2,\alpha}$ -coordinate diffeomorphism Φ_ε in a neighborhood U of the origin with the following properties:

- (i) Φ_ε coincides with the identity map up to the second order at the origin, the restriction $\Phi_\varepsilon|_{f_j(\mathbb{D})}$ is holomorphic with respect to J_{st} and $\Phi_\varepsilon(f_j(\mathbb{D})) = D_j$, $j = 1, 2$;
- (ii) in the new coordinates one has $\Pi = \{w = 0\}$ and S^2 has the form (2.1);
- (iii) $\|\Phi_\varepsilon - id\|_{C^2} < \varepsilon$, where id denotes the identity map.

Step 2: local deformation of the structure. Thus in the new coordinates one may identify the disc $f_j(\mathbb{D})$ with the domain D_j in the axis $\{(z, 0)\}$ and there exists a small perturbation of S^2 satisfying (i), (ii) and such that it has the form (2.2); for simplicity of notations we still denote it by S^2 . The structure $\tilde{J}_\varepsilon := (\Phi_\varepsilon)_*(J)$ coincides with J_{st} at the origin up to the first order. Furthermore, $\tilde{J}_\varepsilon|_{D_j} = J_{st}$. Let $\tilde{A}_\varepsilon := A_{\tilde{J}_\varepsilon}$ be the matrix of the deformation tensor of \tilde{J}_ε , see [10] (the complex matrix of the structure \tilde{J}_ε in the terminology of [20]). Recall that there exists a one-to-one correspondence between an almost complex structure and its deformation tensor, see [20]. In our case $A_{\tilde{J}_\varepsilon}$ vanishes at the origin together with all first order partial derivatives and vanishes on the domains D_j .

Step 3: local deformation of the sphere. Let V be a neighborhood of the origin in \mathbb{C} and $\psi : V \rightarrow \mathbb{R}_+$ be a smooth function, $\psi(0) > 0$, with support compactly contained in V . Let $0 < \delta = \delta(\varepsilon) < \varepsilon$ be small enough. Consider the surface S_ε^2 defined by

$$(5.1) \quad w = z\bar{z} + \gamma Re z^2 - \delta\psi(z).$$

Then S_ε^2 coincides with S^2 outside a small neighborhood of the origin. The real surface $\Pi = \{w = 0\}$ is J_{st} -complex. Its intersection with S_ε^2 is a real curve in the J_{st} -totally real part of S_ε^2 and coincides with the boundaries of D_j outside a neighborhood of the origin. This curve bounds a J_{st} -holomorphic disc \tilde{f}_ε on Π coinciding with $D_1 \cup D_2$ outside a neighborhood of the origin. The family \tilde{f}_ε tends to $D_1 \cup D_2$ from outside as $\varepsilon \rightarrow 0$. Fix a smooth function $\chi(t)$ with $\chi(t) = 0$ for $t < 1$ and $\chi(t) = 1$ for $t > 2$. Consider the matrix $\hat{A}_\varepsilon(z, w) = \chi((|z| + |w|)/\delta_1)\tilde{A}_\varepsilon(Z)$. Here $0 < \delta_1 < \varepsilon$ is small enough. Then \hat{A}_ε tends to

\tilde{A}_ε in the C^1 norm as $\delta_1 \rightarrow 0$ (recall that \tilde{A}_ε vanishes at the origin up to the first order). Furthermore, \hat{A}_ε vanishes near the origin, coincides with \tilde{A}_ε outside a neighborhood of the origin and in the sectors D_j where \tilde{A}_ε vanishes. Fixing $\delta \ll \delta_1$ we obtain that the disc \tilde{f}_ε is holomorphic with respect to the almost complex structure \hat{J}_ε defined by the matrix \hat{A}_ε . Hence the disc $f_\varepsilon = (\Phi_\varepsilon)^{-1}(\tilde{f}_\varepsilon)$ and the structure $J_\varepsilon := (\Phi_\varepsilon)^{-1}(\hat{J}_\varepsilon)$ satisfy the assertion of the lemma. Q.E.D.

Remark. In the above proof we slightly perturbed the sphere S^2 and the almost complex structure J near a good hyperbolic point. Since the boundary ∂M is strictly Levi convex near every hyperbolic point, it remains strictly Levi convex after this C^1 -perturbation of J .

We assume that the point p is a positive hyperbolic point and that the two hyperbolic discs f_1 and f_2 are limits of two maximal positive families (f_1^t) and (f_2^t) of Bishop discs with Maslov indices equal to 0. Applying Lemma 5.1 we obtain a disc f_ε approaching the union $f_1(\mathbb{D}) \cup f_2(\mathbb{D})$ from outside as ε tends to 0.

Lemma 5.2. *The Maslov index of the disc f_ε is equal to 0.*

Proof. Fix t large enough and consider the real oriented surface E which is an open piece of S^2 bounded by the curves $f_1^t(\partial\mathbb{D})$, $f_2^t(\partial\mathbb{D})$ and $f_\varepsilon(\partial\mathbb{D})$. We choose on these curves the orientation induced by the orientation of E and denote the obtained loops by γ_1 , γ_2 and γ_3 respectively. Thus, the orientations of γ_j , $j = 1, 2$ are opposite to the orientations induced by the discs f_j^t , $j = 1, 2$ and the orientation of γ_3 coincides with the orientation induced by f_ε . The boundary of E is totally real with respect to the almost complex structure J_ε from Lemma 5.1 and E contains one positive hyperbolic point p . So $I_+(E) = -1$ and $I_-(E) = 0$. The Maslov indices of f_j^t , $j = 1, 2$ are equal to 0. Hence $I_E(f_j^t|\partial\mathbb{D}) = 1$, $j = 1, 2$ and the sum of the indices $I_E(\gamma_j)$ is equal to -2 . Then by Proposition 3.5 we obtain $I_E(\gamma_3) = +1$ and the Maslov index of f_ε is equal to 0. Q.E.D.

In Proposition 5.1 and Lemma 5.2 we moved two hyperbolic discs f_1 and f_2 to a single disc which is "above" the hyperbolic point creating a family of discs approaching the hyperbolic point from outside; this family bifurcates in the initial hyperbolic discs f_j , $j = 1, 2$. This construction is useful when f_j are obtained as limits of two families of Bishop discs approaching a hyperbolic point from inside. Of course, one can reverse this dynamics of families of Bishop discs and adapt it for the case of outside approach. Since we do not need it here, we drop the details.

5.2. Hyperbolic chains. We considered above the case where hyperbolic discs touch exactly one hyperbolic point. It is easy to see that this construction admits a generalisation to the case where discs touch several hyperbolic points. More precisely, let f_j , $j = 1, \dots, k$, be hyperbolic discs and let $\mathcal{D} = \cup_{j=1}^k f_j(\mathbb{D})$ be their union. We will use the notation $G_j = f_j(\mathbb{D})$; we stress out that the discs G_j forming a chain are closed.

Suppose that $\overline{\mathcal{D}}$ is connected. We know that every disc touches a good hyperbolic point with a good asymptotic approach and by the uniqueness principle, if two discs touch the same hyperbolic point, they approach it from opposite regions. Such family of discs (a nodal curve) is called in [2] a *hyperbolic chain*. A chain \mathcal{D} is called *saturated* if for every hyperbolic point in $\overline{\mathcal{D}}$ both approach regions are filled by discs from that chain.

Lemma 5.3. *If a chain \mathcal{D} containing k discs is not saturated, then it contains at least k hyperbolic points.*

Proof. We proceed by induction in k . Consider the case where $k = 2$. If \mathcal{D} contains only one hyperbolic point, by Section 2 the discs G_j , $j = 1, 2$, fill opposite admissible regions for that point and the chain is saturated. Suppose now that the assertion of the Lemma holds for every chain containing at most $k - 1$ discs.

Consider a chain \mathcal{D} containing k discs and denote by m the number of its hyperbolic points. Since the chain is not saturated, there exists a hyperbolic point for which just one admissible region is filled, say, by the disc G_k . Removing G_k , we obtain a new chain \mathcal{D}' containing $k - 1$ discs and $m - 1$ hyperbolic points. This new chain can not be saturated since the disc G_k is different from the discs forming \mathcal{D}' and approaches at least one point of the chain \mathcal{D}' . Applying the induction assumption, we conclude. Q.E.D.

Furthermore, in general a chain $\overline{\mathcal{D}}$ can be non-simply connected. For example, one may think about a "triangle" formed by three hyperbolic discs intersecting in three "vertices" that are the hyperbolic points. Then a closed path formed by the diameters of these discs with the ends at the hyperbolic points, is not homotopic to a point in \mathcal{D} .

Lemma 5.4. *A non-simply connected chain consisting of k discs contains at least k hyperbolic points.*

Proof. Again we proceed by induction in k . Consider the case $k = 2$. If \mathcal{D} contains only one hyperbolic point, it is formed by two discs glued together at this point and it is simply connected. Suppose that the assertion of the Lemma holds for all chains containing at most $k - 1$ discs.

Consider a chain \mathcal{D} formed by k discs and containing m hyperbolic points. If $m < k$, then at least one disc, say G_k , contains only one hyperbolic point p . Remove that disc from the chain obtaining the new chain \mathcal{D}' . Since by the previous Lemma the chain \mathcal{D} is saturated, the hyperbolic point p remains in the chain \mathcal{D}' and belongs to a single disc, say G_{k-1} . Slightly deforming the disc G_{k-1} and the almost complex structure J near p , we obtain a new chain (for the deformed structure) that does not contain p ; hence it contains $m - 1$ hyperbolic points and it is simply connected by assumption. Hence \mathcal{D}' also is simply connected. But \mathcal{D} is obtained by gluing the disc G_k at a single point p , so it is simply connected. Q.E.D.

In the previous Subsection we proved the existence of a deformation for saturated simply connected hyperbolic chains containing two discs and one hyperbolic point. This construction immediately generalises to the case of saturated simply connected chains containing k discs. We give details in the next section.

6. FILLING SPHERES

Now we prove Theorem 1.1. We proceed by induction on the number N of hyperbolic points in S^2 .

The case $N = 0$ is treated in [10].

Consider the model case $N = 1$: S^2 contains exactly one hyperbolic point H . This point may be assumed positive changing the orientation of S^2 if necessary. Then S^2 has three elliptic points E_j , $j = 1, 2, 3$, and necessarily two of them are positive because $I_+(S^2) = I_-(S^2) = 1$. Denote by E_1 and E_2 the positive elliptic points and by E_3 the negative one. Let (f_j^t) be the maximal positive families of Bishop discs generated by E_j , $j = 1, 2, 3$. According to Lemma 4.8, the families (f_j^t) , $j = 1, 2$ end up into two hyperbolic discs f_1 and f_2 . We point out that the families (f_j^t) , $j = 1, 2$ can not approach the negative elliptic point E_3 : in that case they would touch the discs from the family (f_3^t) and so would coincide with

these discs by the uniqueness principle from [10], which is impossible. By Lemma 2.7, $f_1(\mathbb{D})$ and $f_2(\mathbb{D})$ approach H from two opposite regions. Applying Proposition 5.1 we obtain a family of discs (f_ε) approaching H from outside; they are holomorphic with respect to a structure J_ε which is a small perturbation of J near H . Let now \tilde{f}_3^t be the family of J_ε -holomorphic Bishop discs generated by E_3 . By the uniqueness principle from [10] the disc f_ε is necessarily contained in the family (\tilde{f}_3^t) . Passing to the limit as $\varepsilon \rightarrow 0$, we conclude that S^2 (after a small generic perturbation) is filled by boundaries of J -holomorphic discs.

Consider the case $N \geq 2$. We have $I_+(S^2) = I_-(S^2) = 1$. Let E_1, \dots, E_d be the positive elliptic points. Consider the positive families of Bishop discs (f_j^t) generated by E_j .

Case 1. The maximal families (f_j^t) touch only positive hyperbolic points. Every family (f_j^t) approaches a positive hyperbolic point from inside and fills one approach region. Denote by f_j , $j = 1, \dots, d$, the limit hyperbolic disc for every family. We can regroup these discs to a finite family of disjoint hyperbolic chains \mathcal{D}_l . Since the number of positive hyperbolic points is less than d , one of the chains, say \mathcal{D}_1 , contains more discs than hyperbolic points. Let n and m be respectively the numbers of the discs and the hyperbolic points in \mathcal{D}_1 . Since $m < n$, this chain is saturated and simply connected. Applying to this chain the deformation construction from the previous section (Proposition 5.1), we swept an open subset of S^2 containing the hyperbolic points from \mathcal{D}_1 by the boundaries of discs holomorphic with respect to an ε -perturbed almost complex structure. We extend these families past the hyperbolic points; since the chain saturated and simply connected, we obtain $(n-m)$ positive families of Bishop discs of the Maslov index 0. Thus we removed the same number of discs and hyperbolic points and may proceed by induction. Finally we obtain a filling of S^2 by boundaries of discs holomorphic with respect to an almost complex structure J_ε obtained from J by an ε -perturbation near every hyperbolic point. Passing to the limit as $\varepsilon \rightarrow 0$, we conclude.

Case 2. There are negative hyperbolic points which are limits of the families (f_j^t) . Once again we regroup the limit hyperbolic discs f_j into disjoint chains \mathcal{D}_j . Since a positive family of discs approaches a negative hyperbolic point from outside, it fills both approach regions by Section 4 and the limit chains are saturated at every negative hyperbolic point; furthermore, every negative hyperbolic point attracts precisely one positive family of discs. Next we may apply a deformation construction similarly to Proposition 5.1 at every negative hyperbolic point and slightly deform every chain near such a point; we also suitably deform the almost complex structure so that the new chains remain holomorphic. The new chains contain only positive hyperbolic points and by the previous argument, one of these chains is saturated and simply connected. Since such a chain is obtained by a small deformation of an old one, we conclude that one of the initial chains, say \mathcal{D}_1 , is saturated and simply connected. Then we extend this chain through positive hyperbolic points by Proposition 5.1 replacing two positive families and at least one positive hyperbolic point by a family of discs of Maslov index 0. If the above chains contain only negative hyperbolic points, we consider families of Bishop's discs starting from negative hyperbolic points. Such a family fills both admissible regions near a positive hyperbolic point. Hence in every chain obtained from negative families of discs every positive hyperbolic point attracts precisely one family of discs. Since $I(S) = 2$, at least one of these chains contains a negative hyperbolic point and we apply the above argument extending the chain through that point by Proposition 5.1. Thus, in any case we remove at least one hyperbolic point and two families of discs

replacing them by a single family. By the induction assumption, we conclude the proof of Theorem 1.1.

Thus, we obtain a real hypersurface Γ with boundary S^2 . By construction, Γ is a real smooth hypersurface foliated by a 1-parameter family of J -holomorphic discs; it is obvious from the above construction that Γ is diffeomorphic to the 3-ball.

7. CONCLUDING REMARKS

In this section we compare Theorem 1.1 with related results. We do not discuss numerous applications of this Theorem to symplectic and contact geometry; see, for instance [5].

0. A true breakthrough in the study of filling of 2-spheres in presence of hyperbolic points was done by E.Bedford-W.Klingenberg [2]. They consider the case of spheres with elliptic and good hyperbolic points generically embedded into a strictly pseudoconvex hypersurface in \mathbb{C}^2 . This work remains an important reference in the subject.

1. In the interesting and important work by R.Hind [11] the following situation is considered. Let (M, J, ω) denote a symplectic manifold with a tame almost complex structure containing no holomorphic spheres of negative self-intersection. Let Ω be a smoothly bounded domain with Levi convex boundary $\partial\Omega$. Suppose that $\partial\Omega$ is not the cartesian product of a holomorphic sphere with the circle S^1 and let S^2 be a real 2-sphere generically embedded into $\partial\Omega$. Suppose that J is integrable in a neighborhood of every hyperbolic point of S^2 . Then, if necessarily after a C^2 perturbation in a neighborhood of the complex points, there exists a filling of S^2 by boundaries of holomorphic discs. The work contains a detailed description of the properties of this filling. Admitting that the result holds in the case where $\partial\Omega$ is strictly Levi convex, the author uses Y.Eliashberg-W.Thurston's theorem [6] on approximation of Levi convex boundaries by strictly Levi convex ones. However, in the almost complex setting the corresponding result on filling in the strictly Levi convex case was never proved (it was announced by Y.Eliashberg in [5]).

The present paper fixes that gap. In fact, we prove much more since Theorem 1.1 is established in the Levi convex case under the assumption that there are no non-constant holomorphic discs in the boundary. This is the main case considered in the work [11] since the case where the boundary contains holomorphic discs can be reduced to the previous one by approximation of the boundary, see [11]. We also point out that many other technical simplifications of R.Hind's work are given in [10] using exhaustion plurisubharmonic functions.

Note that in general, the condition of the Levi convexity cannot be dropped. It was first observed by Y.Eliashberg [5]; later J.Fornaess-D.Ma [7] constructed an explicit example.

2. N.Kruzhilin proved in [14] the existence of a filling of a two-sphere generically embedded into a strictly Levi convex hypersurface in \mathbb{C}^2 (with the standard complex structure) without any assumption that the hyperbolic points are good. His description of the boundary behaviour of hyperbolic discs is substantially more complicated. It is natural to think that a combination of his techniques with the methods of the present work will allow to obtain Theorem 1.1 without assuming that the hyperbolic points are good. Another (more general) open question is if Theorem 1.1 remains true when the almost complex structure J is not supposed to be integrable near hyperbolic points. However, we must point out that for many applications the condition of the integrability of J near hyperbolic points and the assumption that these points are good are not restrictive and naturally hold. Indeed, by a small perturbation of an almost complex structure a given two-sphere can be lead into

a position where the assumptions of Theorem 1.1 hold; the integrability of the structure near complex points also often can be assumed. Thus, Theorem 1.1 is in general sufficient for filling a single sphere. So it is quite possible that the necessary technical difficulties to answer the above open questions will not correspond to the impact.

3. As mentioned in [10], the condition that (M, J, ω) contains no non-constant holomorphic spheres may be weakened. We do not develop this subject here referring to the concluding remarks in [10].

4. Many parts of the proof presented in our article still work in the case where instead of the real sphere S^2 one considers a compact real surface (with or without boundary) contained in a pseudoconvex hypersurface and admitting a finite number of complex points. Here suitable assumptions on the numbers of elliptic and hyperbolic points are necessary. We hope that it will be useful in further applications.

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